Chapter 14
Extensions

EXTENSION 14E1: THE RELATIONSHIP BETWEEN THE MULTIVARIATE AND THE MIXED-MODEL APPROACHES—TWO-WAY WITHIN-SUBJECTS DESIGNS

The multivariate and mixed-model approaches to analyzing data from two-way within-subjects designs relate to one another in a very similar manner to the way they are related for analyzing data from one-way designs, which we discussed in Chapter 13. To make the relationship between the two methods explicit, it is again necessary to work with orthonormal contrasts. Recall that orthonormal contrasts must be orthogonal and that the sum of squared coefficients for an orthonormalized contrast equals 1.0.

To develop the relationship between the two approaches, we again consider the data in Table 14.4 in the book. Recall that we formed five D variables to test the two main effects and the interaction. When we expressed the coefficients of these variables as integers, the equations for the five D variables were as follows:

$$D_{1i} = -1Y_{1i} + 0Y_{2i} + 1Y_{3i} - 1Y_{4i} + 0Y_{5i} + 1Y_{6i}, \qquad \text{(14.20, repeated)}$$

$$D_{2i} = 1Y_{1i} - 2Y_{2i} + 1Y_{3i} + 1Y_{4i} - 2Y_{5i} + 1Y_{6i}, \qquad \text{(14.21, repeated)}$$

$$D_{3i} = -1Y_{1i} - 1Y_{2i} - 1Y_{3i} + 1Y_{4i} + 1Y_{5i} + 1Y_{6i}, \qquad \text{(14.22, repeated)}$$

$$D_{4i} = 1Y_{1i} + 0Y_{2i} - 1Y_{3i} - 1Y_{4i} + 0Y_{5i} + 1Y_{6i}, \qquad \text{(14.23, repeated)}$$

$$D_{5i} = -1Y_{1i} + 2Y_{2i} - 1Y_{3i} + 1Y_{4i} - 2Y_{5i} + 1Y_{6i}. \qquad \text{(14.24, repeated)}$$

It can be shown that these contrasts are all orthogonal to each other; that is, they form an orthogonal set.[1] Thus all that remains to be done is to normalize the coefficients of each contrast. As in Chapter 13, this is accomplished by dividing each nonnormalized coefficient by the square root of the sum of squared coefficients for that particular contrast. For example, because the sum of squared coefficients for D_1 is 4, we need to divide each nonnormalized coefficient by 2 (the square root of 4). Carrying out this process

TABLE 14E1.1
E*(F) AND E*(R) MATRICES FOR A MAIN EFFECT, B MAIN EFFECT, AND $A \times B$ INTERACTION

Effect		
A	$\mathbf{E}^*(\mathbf{F})$ $\begin{bmatrix} 38{,}160.0 & -1{,}870.6 \\ -1{,}870.6 & 25{,}920.0 \end{bmatrix}$	$\mathbf{E}^*(\mathbf{R})$ $\begin{bmatrix} 320{,}400.0 & -44{,}686.9 \\ -44{,}686.9 & 33{,}600.0 \end{bmatrix}$
B	[76,140]	[361,800]
$A \times B$	$\begin{bmatrix} 11{,}160.0 & 3{,}325.5 \\ 3{,}325.5 & 9{,}720.0 \end{bmatrix}$	$\begin{bmatrix} 115{,}200.0 & -7{,}274.6 \\ -7{,}274.6 & 10{,}800.0 \end{bmatrix}$

for all five D variables results in the following orthonormal set of D^* variables:

$$\begin{aligned}
D_{1i}^* &= -.5Y_{1i} + 0Y_{2i} + .5Y_{3i} - .5Y_{4i} + 0Y_{5i} + .5Y_{6i}\,, \\
D_{2i}^* &= .2887Y_{1i} - .5774Y_{2i} + .2887Y_{3l} + .2887Y_{4l} - .5774Y_{5i} + .2887Y_{6i}\,, \\
D_{3i}^* &= -.4082Y_{1i} - .4082Y_{2i} - .4082Y_{3i} + .4082Y_{4i} + .4082Y_{5i} + .4082Y_{6i}\,, \\
D_{4i}^* &= .5Y_{1i} + 0Y_{2i} - .5Y_{3i} - .5Y_{4i} + 0Y_{5i} + .5Y_{6i}\,, \\
D_{5i}^* &= -.2887Y_{1i} + .5774Y_{2i} - .2887Y_{3i} + .2887Y_{4i} - .5774Y_{5i} + .2887Y_{6i}\,.
\end{aligned}$$

Remember that D_1^* and D_2^* represent the angle main effect, D_3^* represents the noise main effect, and D_4^* and D_5^* represent the interaction. For each test, there is a full matrix, denoted **E*(F)**, and a restricted matrix, denoted **E*(R)**. Computation of these matrices follows the same principles used in Chapter 13, so we do not bother with computational details here. Instead, we simply refer you to Table 14E1.1, which presents three **E*(F)** and three **E*(R)** matrices, one for each of the three effects being tested.

Comparing Table 14E1.1 to Table 12.5 shows that the same relationship holds here between the sum of the diagonal elements of an **E*** matrix in the multivariate approach and a sum-of-squares term in the mixed-model approach. For example, the sum of the two diagonal elements of **E*(F)** for the A main effect equals 64,080, which is equal to $SS_{\mathrm{A\times S}}$ in the mixed-model approach. The sum of the two diagonal elements of **E*(R)** for the A main effect equals 354,000. Subtracting 64,080 from 354,000 yields 289,920, which is the value of SS_A in the mixed-model approach. The same type of equality holds for the B and the $A \times B$ effects. As a result, the mixed-model F for an effect can again be written in terms of the multivariate matrices as

$$F = \frac{\{tr(\mathbf{E}^*(\mathbf{R})) - tr(\mathbf{E}^*(\mathbf{F}))\} / df_{\text{effect}}}{tr(\mathbf{E}^*(\mathbf{F})) / (n-1)df_{\text{effect}}}\,, \tag{14E1.1}$$

where tr(**E*(R)**) and tr(**E*(F)**) denote the trace (i.e., sum of diagonal elements) of the restricted and full matrices for the effect being tested. Notice that Equation 14E1.1 is a straightforward generalization of Equation 13.34, which we developed for a one-way within-subjects design:

$$F = \frac{(tr(\mathbf{E}^*(\mathbf{R})) - tr(\mathbf{E}^*(\mathbf{F}))) / (a-1)}{tr(\mathbf{E}^*(\mathbf{F})) / (n-1)(a-1)}. \qquad \text{(13.34, repeated)}$$

As in the one-way design, the mixed-model F test differs from the multivariate F test because the mixed-model F test is based on an assumption of sphericity. If the sphericity assumption is met, the population values of the off-diagonal elements of an **E*(F)** matrix are all zero. In addition, if sphericity holds, the population values of the diagonal elements of **E*(F)** are all equal to one another; thus, the sample mean of these values (i.e., $tr(\mathbf{E}^*(\mathbf{F}))/df_{\text{effect}}$) is a good estimate of the single underlying population value. As we discussed in Chapter 12, sphericity may be met for some effects and yet fail for other effects, even in the same study. For example, the sphericity assumption is necessarily true for the B effect in our study because the B factor has only two levels. As a result, we only needed to form one D variable to capture the B main effect, and there are no off-diagonal elements in **E*(F)** for the B main effect (see Table 14E1.1). Also, there is only one diagonal element of **E*(F)**; thus, equality of all diagonal elements need not be a concern. Not only is there no need to assume sphericity for the B effect here but also the mixed-model and multivariate approaches yielded exactly the same F value for the B main effect in our data. With both approaches, the F value was 33.77, with 1 and 9 *df.* Such an equality always occurs for all single degree of freedom effects, as long as $MS_{\text{effect}\times\text{S}}$ is used as the error term in the mixed-model approach.

It is also important to realize that the test of the B main effect in our example is valid even if compound symmetry fails to hold for the 6 × 6 matrix that would result from correlating scores in the six different conditions. Recall that compound symmetry requires that all correlations be equal to one another in the population. However, we have just argued that the sphericity assumption is always met for an effect with only 1 *df* (as long as $MS_{\text{effect}\times\text{S}}$ is used as the error term), so sphericity and compound symmetry are different assumptions. It can be shown that compound symmetry implies sphericity—that is, if the compound symmetry assumption is met, the sphericity assumption is also met. However, the reverse is not always true, because it is possible for sphericity to hold in the absence of compound symmetry.

EXTENSION 14E2: THE RELATIONSHIP BETWEEN THE MULTIVARIATE AND THE MIXED-MODEL APPROACHES: SPLIT-PLOT DESIGNS

The multivariate and mixed-model approaches for analyzing data from a split-plot design relate to one another as they do for other designs. As before, we do not provide a mathematical proof of this relationship, but instead demonstrate it empirically for our data.

An appropriate reminder at this point is that the multivariate and mixed-model approaches yield identical results for testing between-subjects effects. In essence, if there is only one score per participant entered into a particular test of significance, the two approaches produce equivalent results. Thus our comparison of the two approaches concerns itself with the main effect of the within-subjects factor and with the interaction between the two factors.

Recall that we formed two D variables in our example to perform the multivariate test. The first D variable represented the linear trend for angle and was defined as

$$D_{1ij} = -1Y_{1ij} + 0Y_{2ij} + 1Y_{3ij}.$$

The second D variable, which represented the quadratic trend for angle, was defined as

$$D_{2ij} = 1Y_{1ij} - 2Y_{2ij} + 1Y_{3ij}.$$

To compare the two approaches, we must normalize the coefficients of these contrast variables. Notice that the linear and quadratic trends are already orthogonal to one another, so we need not worry further about this requirement. As usual, each nonnormalized coefficient must be divided by the square root of the sum of squared coefficients for that particular contrast. Carrying out this process for the linear and quadratic trend variables yields

$$D_{1ij}^{*} = -.7071Y_{1ij} + 0Y_{2ij} + .7071Y_{3ij}\ ,$$
$$D_{2ij}^{*} = .4082Y_{1ij} - .8164Y_{2ij} + .4082Y_{3ij}\ .$$

We could now perform the multivariate test on D_1^{*} and D_2^{*} by calculating a full matrix **E*(F)** and two restricted matrices **E*(R)**, one for the B main effect and one for the $A \times B$ interaction. Although such a procedure would produce the desired results, it is much simpler to work directly from the matrices we already calculated for the nonnormalized D_1 and D_2 variables. Earlier in the chapter, we found that the full and restricted matrices for D_1 and D_2 were given by

$$\mathbf{E}(\mathbf{F}) = \begin{bmatrix} 67{,}050 & -9{,}090 \\ -9{,}090 & 125{,}370 \end{bmatrix},$$

$$\mathbf{E_B}(\mathbf{R}) = \begin{bmatrix} 936{,}495 & -52{,}875 \\ -52{,}875 & 127{,}575 \end{bmatrix},$$

$$\mathbf{E_{A\times B}}(\mathbf{R}) = \begin{bmatrix} 99{,}855 & 21{,}285 \\ 21{,}285 & 153{,}495 \end{bmatrix},$$

where $\mathbf{E_B(R)}$ is the restricted matrix for the B main effect and $\mathbf{E_{A \times B}(R)}$ is the restricted matrix for the $A \times B$ interaction. We can compute the **E*** matrices for the normalized variables by realizing that for each participant,

$$D_{1ij}^{*} = .7071D_{1ij}\ ,$$
$$D_{2ij}^{*} = .4082D_{2ij}.$$

As a result,

$$(D_{1ij}^{*})^2 = .5(D_{1ij})^2\ ,$$

$$(D_{2ij}^{*})^2 = .1667(D_{2ij})^2\ ,$$

$$D_{1ij}^{*}D_{2ij}^{*} = .2887D_{1ij}\ \mathrm{D}_{2ij}\ .$$

It then follows that the row 1, column 1 element of each **E*** matrix equals .5 times the corresponding element of each **E** matrix. Similarly, the row 2, column 2 element of each **E*** matrix equals .1667 times the corresponding element of each **E** matrix. Finally, the row 1, column 2 and row 2, column 1 elements of each **E*** matrix equals .2887 times the corresponding elements of each **E** matrix. Carrying out the necessary multiplication results in the following **E*** matrices:

$$\mathbf{E}^*(\mathbf{F}) = \begin{bmatrix} 33,525.00 & -2,624.06 \\ -2,624.06 & 20,895.00 \end{bmatrix},$$

$$\mathbf{E}^*_{\mathbf{B}}(\mathbf{R}) = \begin{bmatrix} 468,247.50 & -15,263.70 \\ -15,263.70 & 21,262.50 \end{bmatrix},$$

$$\mathbf{E}^*_{\mathbf{A\times B}}(\mathbf{R}) = \begin{bmatrix} 49,927.50 & 6,144.45 \\ 6,144.45 & 25,582.50 \end{bmatrix}.$$

We can now consider the relationship between these **E*** matrices and three sums of squares in the mixed-model approach: $SS_{\mathrm{B\times S/A}}$, SS_{B}, and $SS_{\mathrm{A\times B}}$. The sum of the two diagonal elements of **E*(F)** equals 54,420, which we saw in Chapter 12 is $SS_{\mathrm{B\times S/A}}$ for our data. Remembering that the sum of the diagonal elements of a matrix is its trace, which is abbreviated *tr*, enables us to write

$$SS_{\mathrm{B\times S/A}} = tr\,(\mathbf{E}^*\,(\mathbf{F})).$$

The relationships involving SS_{B} and $SS_{\mathrm{A\times B}}$ are similarly given by

$$SS_{\mathrm{B}} = tr(\mathbf{E}^*_{\mathbf{B}}(\mathbf{R})) - tr(\mathbf{E}^*(\mathbf{F}))\,,$$
$$SS_{\mathrm{A\times B}} = tr(\mathbf{E}^*_{\mathrm{A\times B}}(\mathbf{R})) - tr(\mathbf{E}^*(\mathbf{F}))\,.$$

As a result, the mixed-model F tests for B and $A \times B$ can be written as

$$F = \frac{\{tr(\mathbf{E}^*_{\mathbf{B}}(\mathbf{R})) - tr(\mathbf{E}^*(\mathbf{F}))\} / (b-1)}{tr(\mathbf{E}^*(\mathbf{F})) / (N-a)(b-1)} \tag{14E2.1}$$

for the B main effect, and

$$F = \frac{\{tr(\mathbf{E}^*_{\mathbf{A\times B}}(\mathbf{R})) - tr(\mathbf{E}^*(\mathbf{F}))\} / (a-1)(b-1)}{tr(\mathbf{E}^*(\mathbf{F})) / (N-a)(b-1)} \tag{14E2.2}$$

for the $A \times B$ interaction.

The practical implication of Equations 14E2.1 and 14E2.2 is that once again the multivariate approach is sensitive to all elements of the **E*** matrices, whereas the mixed-model approach ignores the off-diagonal elements. The reason, as before, is that if the sphericity assumption required by the mixed-model approach is met, the population values of the off-diagonal elements of the **E*(F)** matrix are all zero. In addition, if sphericity holds, the population values of the diagonal elements of **E*(F)** are all equal to one another so that the sample mean of these values—that is, $tr(\boldsymbol{E}^*(\boldsymbol{F}))/(b-1)$—is a good estimate of the single underlying population value. However, if sphericity fails to hold, the mixed-model approach suffers from an inflated Type I error rate, unless ε adjustments are applied to the degrees of freedom of the critical value.

Also notice that the trace of the **E*(F)** matrix forms the denominator sum of squares for testing both the B main effect and the $A \times B$ interaction. For this reason, the sphericity assumption is either met for both effects or it fails for both. When there is only one within-subjects factor, there is only one matrix for which the sphericity assumption is an issue. If, however, there were a second within-subjects factor in the design, we would need to consider additional matrices, just as we did earlier in the chapter for factorial within-subjects designs.

NOTE

1. You should be able to convince yourself that these contrasts are indeed orthogonal by applying the test for orthogonality that was presented in Chapter 4.